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Deformation quantization of noncommutative quantum mechanics*

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Abstract: The Wigner phase-space distribution function provides the basis for Moyal's deformation quantization alternative to the more conventional Hilbert space and path integral quantization. In this paper, we investigate the basic aspects of deformation quantization for noncommutative quantum mechanics (NCQM). We first prove some general relations of the Weyl correspondence in non-commuting phase-space. Then we derive explicit form of the Wigner Function (WF) for NCQM starting from fundamental principle of the Weyl correspondence, and show that it satisfies a generalized $*$-genvalue equation. We also demonstrate that the new WFs possess orthonormality and completeness, so they can be used as a basis to expand all phase-space functions. Some example is discussed to support our results.

Keywords: Statistical Methods, Non-Commutative Geometry.

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1. Introduction

Recently a large amount of research work has been devoted to the study of quantum mechanics on noncommutative space [1]–[6]. For two-dimensional plane it can be characterized by the following brackets

$$[\hat{x}_1,\, \hat{x}_2] = i\theta\,, \qquad [\hat{x}_i,\, \hat{p}_j] = i\hbar\delta_{i,j}\,, \qquad [\hat{p}_1,\, \hat{p}_2] = 0\,, \qquad (i,j=1,2)\,, \tag{1.1}$$

where θ is a parameter characterizing the spacial noncommutativity, and by the hamiltonian $\hat{H} = \frac{1}{2m}(\hat{p}_1^2+\hat{p}_2^2)+V(\hat{x}_1,\hat{x}_2)$ with some potential $V(\hat{x}_1,\hat{x}_2)$. Since the coordinate operators $\hat{x}_1,\hat{x}_2$ do not commute each other, they have no common eigenstates, so it causes no way to build the practical useful coordinate representations and number representations like in the ordinary quantum mechanics. This brings difficulties into the study of NCQM according to the standard Schrödinger-Heisenberg-Dirac formulation in Hilbert space.

It is well known that there is a third presentation of quantum mechanics besides the well-known Hilbert space and path integral quantization, that is, the deformation quantization in the phase-space [7]–[11], which based on Wigner's (1932) quasi-distribution function and Weyl's (1927) correspondence between quantum mechanical operators and ordinary c-number phase-space functions. The crucial composition structure of these c-number phase-space functions, which relies on the $*_\hbar$-product [12]

$$*_\hbar = \exp\left(\frac{i\hbar}{2}\left(\overleftarrow{\partial}_{x_1}\overrightarrow{\partial}_{p_1} - \overleftarrow{\partial}_{p_1}\overrightarrow{\partial}_{x_1} + \overleftarrow{\partial}_{x_2}\overrightarrow{\partial}_{p_2} - \overleftarrow{\partial}_{p_2}\overrightarrow{\partial}_{x_2}\right)\right), \tag{1.2}$$

was fully understood by Groenewold (1946), who, together with Moyal (1949), pulled the entire formulation together.

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Here we must clearly understand that NCQM is not at all a different quantum mechanics. It just has a different choice of coordinate variables, which are not commuting, therefore, there is only one kind of deformation quantization. Of course, within NCQM there should be corresponding choice of distribution functions, which must match the basic commutation relations of NCQM, and which should lead to corresponding phase-space kernels.[1]

In this paper we investigate phase-space presentation of the NCQM. Usually phase-space in NCQM can be called the non-commuting phase-space, in contrast with the (commuting) phase-space corresponding to the ordinary quantum mechanics. In order to meet the commutation relations (1.1), we modify the ordinary definition of c-number phase-space kernels of suitably ordered quantum operators. From this new definition, an operator product will correspond to a generalized $*$-composition of these kernels. Then we start from the very beginning principle of Weyl correspondence, derive a new form for WF on noncommutative space and show it satisfies a generalized $*$-genvalue equation. Some properties of the new WFs such as orthonormality and completeness are proved, so that these WFs can be served as a basis to expand all phase-space functions. To demonstrate the essential difference between the new WFs and the old one, we discuss a non-trivial example of noncommutative harmonic oscillator which supports our results.

2. General relations of the Weyl correspondence in non-commuting phase-space

Without loss of generality, we consider only four-dimensional phase-space $(x_i,\, p_i), (i=1,2)$, as the extension to higher dimensions is straightforward. In addition, we mainly address discrete spectra, $E_n,\ n=0,1,2,3,\ldots$, because generalizing to continuous spectra is also trivial.

In NCQM, spacial noncommutativity can be imposed by treating the coordinates as commuting, but introducing $*_\theta$-product instead of the ordinary product between functions $f(\mathbf{x})$ and $g(\mathbf{x})$ [13]

$$f(\mathbf{x}) *_\theta g(\mathbf{x}) \equiv f(\mathbf{x}) \exp\left(\frac{i\theta}{2}(\overleftarrow{\partial}_{x_1}\overrightarrow{\partial}_{x_2} - \overleftarrow{\partial}_{x_2}\overrightarrow{\partial}_{x_1})\right) g(\mathbf{x})\,. \tag{2.1}$$

In this work, we denote the phase-space variables as $\mathbf{x}=(x_1,x_2)$ and $\mathbf{p}=(p_1,p_2)$. In terms of the $*_\theta$-product, Schrödinger equation in NCQM should be written as

$$\hat{H}(\tilde{x}_i,\tilde{p}_i) *_\theta\ \psi(\tilde{x}_i) = E\psi(\tilde{x}_i)\,, \tag{2.2}$$

where the coordinates $\tilde{x}_i$ and their momenta $\tilde{p}_i$ satisfy standard Heisenberg commutative relations

$$[\tilde{x}_i,\, \tilde{p}_j] = i\hbar\,\delta_{i,j}\,, \qquad (i,j=1,2) \tag{2.3}$$

and other commutators are vanishing. In other words, the non-commuting coordinate operators can be expressed in terms of commuting coordinate operators and their momentum

[1] We thank the referee draw our attention to this point of view.

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operators in the form [14]

$$\hat{x}_1 = \tilde{x}_1 - \frac{\theta}{2\hbar}\tilde{p}_2\,, \qquad \hat{x}_2 = \tilde{x}_2 + \frac{\theta}{2\hbar}\tilde{p}_1\,, \qquad \hat{p}_1 = \tilde{p}_1\,, \qquad \hat{p}_2 = \tilde{p}_2\,. \tag{2.4}$$

Originally, in the Weyl correspondence [8], c-number phase-space kernels $a(\mathbf{x}, \mathbf{p})$ of Weyl ordered operators $\hat{A}(\hat{\mathbf{x}}, \hat{\mathbf{p}})$ are defined by

$$a(\mathbf{x}, \mathbf{p}) = \frac{1}{(2\pi)^2} \int d\mathbf{y} e^{-i\mathbf{y}\cdot\mathbf{p}} \left\langle \mathbf{x} + \frac{\hbar}{2}\mathbf{y} \mid \hat{A}(\hat{\mathbf{x}}, \hat{\mathbf{p}}) \mid \mathbf{x} - \frac{\hbar}{2}\mathbf{y} \right\rangle . \tag{2.5}$$

Conversely, the Weyl ordering of these operators is specified through

$$\hat{A}(\hat{\mathbf{x}}, \hat{\mathbf{p}}) = \frac{1}{(2\pi\hbar)^2} \int d\mathbf{x} d\mathbf{p} d\boldsymbol{\tau} d\boldsymbol{\sigma}\, a(\mathbf{x}, \mathbf{p}) \exp\left(\frac{i}{\hbar}\boldsymbol{\tau} \cdot (\hat{\mathbf{p}} - \mathbf{p}) + \frac{i}{\hbar}\boldsymbol{\sigma} \cdot (\hat{\mathbf{x}} - \mathbf{x})\right). \tag{2.6}$$

It is easily to check that, if the coordinate and momentum operators $\hat{x}_i$ and $\hat{p}_i$ in eq. (2.6) satisfy the commutation relations (1.1), then substituting eq. (2.6) into the right-hand side of eq. (2.5) would not give out the left-hand side of eq. (2.5). In fact, instead of $a(\mathbf{x}, \mathbf{p})$, one gets $a(x_1 - \frac{\theta}{2\hbar}p_2,\, x_2 + \frac{\theta}{2\hbar}p_1,\, p_1,\, p_2)$.

Physically, eq. (2.6) requires that an arbitrary operator, regarded as a power series in $\hat{x}_i$ and $\hat{p}_i$, be first ordered in a completely symmetrized expression in $\hat{x}_1$, $\hat{x}_2$, $\hat{p}_1$ and $\hat{p}_2$, by use of commutation relations (1.1). A term with k powers of $\hat{x}_1$, l powers of $\hat{x}_2$, m powers of $\hat{p}_1$ and n powers of $\hat{p}_2$ will be obtained from the coefficient of $\tau_1^m \tau_2^n \sigma_1^k \sigma_2^l$ in the expansion of $(\boldsymbol{\tau} \cdot \hat{\mathbf{p}} + \boldsymbol{\sigma} \cdot \hat{\mathbf{x}})^{k+l+m+n}$. Thus, in order to make eq. (2.6) and eq. (2.5) consistent in NCQM, we must modify the eq. (2.5) as

$$a(\mathbf{x}, \mathbf{p}) = \frac{1}{(2\pi)^2} \int d\mathbf{y} e^{-i\mathbf{y}\cdot\mathbf{p}} \left\langle \mathbf{x} + \frac{\hbar}{2}\mathbf{y} \mid *_\theta \hat{A}(\hat{\mathbf{x}}, \hat{\mathbf{p}}) \mid \mathbf{x} - \frac{\hbar}{2}\mathbf{y} \right\rangle . \tag{2.7}$$

Here we introduce the $*_\theta$-product between bra- and ket-vector and it does not act on $\hat{A}$. Noticing that in eq. (2.7) the state-vector $\mid \mathbf{q}\rangle = \mid q_1, q_2\rangle$ can be considered as eigenstate of the coordinate operators $\tilde{\mathbf{x}}$ with eigenvalues $\mathbf{q} = (q_1, q_2)$, and using eq. (2.4), one has the following useful relations

$$\begin{aligned}
e^{i\lambda\hat{p}_1} \mid q_1, q_2\rangle &= e^{i\lambda\tilde{p}_1} \mid q_1, q_2\rangle = \mid q_1 - \lambda\hbar,\, q_2\rangle\,, \\
e^{i\lambda\hat{p}_2} \mid q_1, q_2\rangle &= e^{i\lambda\tilde{p}_2} \mid q_1, q_2\rangle = \mid q_1,\, q_2 - \lambda\hbar\rangle\,, \\
e^{i\lambda\hat{x}_1} \mid q_1, q_2\rangle &= e^{i\lambda(\tilde{x}_1 - \frac{\theta}{2\hbar}\tilde{p}_2)} \mid q_1, q_2\rangle = e^{i\lambda q_1} \left| q_1,\, q_2 + \frac{\lambda\theta}{2} \right\rangle, \\
e^{i\lambda\hat{x}_2} \mid q_1, q_2\rangle &= e^{i\lambda(\tilde{x}_2 + \frac{\theta}{2\hbar}\tilde{p}_1)} \mid q_1, q_2\rangle = e^{i\lambda q_2} \left| q_1 - \frac{\lambda\,\theta}{2},\, q_2 \right\rangle .
\end{aligned} \tag{2.8}$$

With the aid of

$$\begin{aligned}
&\exp\left(i\sigma_1\hat{x}_1 + i\sigma_2\hat{x}_2 + i\tau_1\hat{p}_1 + i\tau_2\hat{p}_2\right) = \qquad (2.9)\\
&= \exp\left(i\frac{\tau_2}{2}\hat{p}_2\right)\exp\left(i\frac{\tau_1}{2}\hat{p}_1\right)\exp\left(i\frac{\sigma_2}{2}\hat{x}_2\right)\exp\left(i\sigma_1\hat{x}_1\right)\exp\left(i\frac{\sigma_2}{2}\hat{x}_2\right)\exp\left(i\frac{\tau_1}{2}\hat{p}_1\right)\exp\left(i\frac{\tau_2}{2}\hat{p}_2\right),
\end{aligned}$$

and substituting eq. (2.6) into the right-hand side of eq. (2.7), one gets $a(\mathbf{x}, \mathbf{p})$ as expected.

Next an operator product should correspond to a generalized $*$-composition of these classical kernels (2.7)

$$a(\mathbf{x},\mathbf{p}) * b(\mathbf{x},\mathbf{p}) = \frac{1}{(2\pi)^4\hbar^2}\int d\mathbf{y} e^{-i\mathbf{y}\cdot\mathbf{P}}\left\langle \mathbf{x}+\frac{\hbar}{2}\mathbf{y}\middle| \hat{A}(\hat{\mathbf{x}},\hat{\mathbf{p}}) *_\theta \hat{B}(\hat{\mathbf{x}},\hat{\mathbf{p}})\middle| \mathbf{x}-\frac{\hbar}{2}\mathbf{y}\right\rangle, \quad (2.10)$$

where the $*$-product in the left-hand side of eq. (2.10) means $* = *_\hbar *_\theta$. Substituting $\hat{A}(\hat{\mathbf{x}},\hat{\mathbf{p}})$ and $\hat{B}(\hat{\mathbf{x}},\hat{\mathbf{p}})$ according to the expression (2.6) into the right-hand side of (2.10), and using

$$\begin{aligned}
&\left\langle x_1+\frac{\hbar}{2}y_1+\tau_1+\frac{\theta}{2\hbar}\sigma_2,\, x_2+\frac{\hbar}{2}y_2+\tau_2-\frac{\theta}{2\hbar}\sigma_1\right| \times\\
&\times e^{\frac{i}{\hbar}\boldsymbol{\sigma}\cdot\mathbf{x}} *_\theta e^{\frac{i}{\hbar}\boldsymbol{\eta}\cdot\mathbf{x}}\left| x_1-\frac{\hbar}{2}y_1-\lambda_1-\frac{\theta}{2\hbar}\eta_2,\, x_2-\frac{\hbar}{2}y_2-\lambda_2+\frac{\theta}{2\hbar}\eta_1\right\rangle = \qquad (2.11)\\
&= \exp\left(\frac{i}{\hbar}\boldsymbol{\sigma}\cdot\mathbf{x}+\frac{i}{\hbar}\boldsymbol{\eta}\cdot\mathbf{x}\right)\exp\left(\frac{i\theta}{2\hbar^2}(\eta_1\sigma_2-\eta_2\sigma_1)\right)\delta(\hbar y_1+\tau_1+\lambda_1)\delta(\hbar y_2+\tau_2+\lambda_2),
\end{aligned}$$

we can easily obtain the left-hand side of eq. (2.10).

Finally, we neatly work out in detail how the classical kernels $a(\mathbf{x},\mathbf{p})$ and $b(\mathbf{x},\mathbf{p})$ of two operators $\hat{A}$ and $\hat{B}$ must compose to yield the classical kernel of $\hat{A}\hat{B}$

$$\begin{aligned}
\hat{A}(\hat{\mathbf{x}},\hat{\mathbf{p}})\hat{B}(\hat{\mathbf{x}},\hat{\mathbf{p}}) &= \frac{1}{(2\pi\hbar)^8}\int d\boldsymbol{\tau} d\boldsymbol{\sigma} d\boldsymbol{\tau}' d\boldsymbol{\sigma}' d\mathbf{x}' d\mathbf{p}' d\mathbf{x}'' d\mathbf{p}'' a(\mathbf{x}',\mathbf{p}') b(\mathbf{x}'',\mathbf{p}'') \times\\
&\quad \times \exp\left(\frac{i}{\hbar}\big(\boldsymbol{\tau}\cdot(\hat{\mathbf{p}}-\mathbf{p}')+\boldsymbol{\sigma}\cdot(\hat{\mathbf{x}}-\mathbf{x}')\big)\right)\times\\
&\quad \times \exp\left(\frac{i}{\hbar}\big(\boldsymbol{\tau}'\cdot(\hat{\mathbf{p}}-\mathbf{p}'')+\boldsymbol{\sigma}'\cdot(\hat{\mathbf{x}}-\mathbf{x}'')\big)\right)\\
&= \frac{1}{(2\pi\hbar)^8}\int d\boldsymbol{\tau} d\boldsymbol{\sigma} d\boldsymbol{\tau}' d\boldsymbol{\sigma}' d\mathbf{x}' d\mathbf{p}' d\mathbf{x}'' d\mathbf{p}'' a(\mathbf{x}',\mathbf{p}') b(\mathbf{x}'',\mathbf{p}'') \times\\
&\quad \times \exp\left(\frac{i}{\hbar}\Big(-\boldsymbol{\tau}\cdot\mathbf{p}'-\boldsymbol{\sigma}\cdot\mathbf{x}'-\tau'\cdot\mathbf{p}''-\boldsymbol{\sigma}'\cdot\mathbf{x}''+\frac{1}{2}(\boldsymbol{\tau}\cdot\boldsymbol{\sigma}'-\boldsymbol{\sigma}\cdot\boldsymbol{\tau}')\Big)\right)\times\\
&\quad \times \exp\left(\frac{i}{\hbar}\big((\boldsymbol{\tau}+\boldsymbol{\tau}')\cdot\hat{\mathbf{p}}+(\boldsymbol{\sigma}+\boldsymbol{\sigma}')\cdot\hat{\mathbf{x}}\big)\right)\exp\left(\frac{i\theta}{2\hbar^2}(\sigma_2\sigma_1'-\sigma_1\sigma_2')\right). \quad (2.12)
\end{aligned}$$

Changing integration variables to

$$\boldsymbol{\tau}=\boldsymbol{\xi}-2(\mathbf{x}-\mathbf{x}')\,,\qquad \boldsymbol{\tau}'=2(\mathbf{x}-\mathbf{x}')\,,\qquad \boldsymbol{\sigma}=\boldsymbol{\eta}-2(\mathbf{p}'-\mathbf{p})\,,\qquad \boldsymbol{\sigma}'=2(\mathbf{p}'-\mathbf{p})\,, \quad (2.13)$$

we can reduces the above integral to

$$\hat{A}\hat{B} = \frac{1}{(2\pi\hbar)^4}\int d\boldsymbol{\xi} d\boldsymbol{\eta} d\mathbf{x} d\mathbf{p} \exp\left(\frac{i}{\hbar}\Big(\boldsymbol{\xi}\cdot(\hat{\mathbf{p}}-\mathbf{p})+\boldsymbol{\eta}\cdot(\hat{\mathbf{x}}-\mathbf{x})\Big)\right) a(\mathbf{x},\mathbf{p}) * b(\mathbf{x},\mathbf{p})\,. \quad (2.14)$$

This fundamental isomorphism of operator products to associative strings of star multiplications enables us to build the formulation of NCQM on the non-commuting phase-space. In short words, in this section, with the new c-number phase-space kernels eq. (2.7) of Weyl ordered operators, we obtain the entire correspondence relations between the operators in NCQM and their classical phase-space kernels, which provide a solid base of building a formulation of deformation quantization for NCQM. Otherwise, based on the original definition (2.5) of the classical kernel, one cannot construct a consistent formulation to quantize the NCQM on the non-commuting phase-space.

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3. WF on the noncommutative space

As a distribution function of quantum phase-space, WF $f(\mathbf{x},\mathbf{p})$ corresponds to the Weyl ordering of the coordinate $\hat{\mathbf{x}}$ and momentum $\hat{\mathbf{p}}$ and can be obtained from the following equation [15]

$$\mathrm{Tr}\left(\hat{\rho}(\hat{\mathbf{x}},\hat{\mathbf{p}})e^{i\boldsymbol{\xi}\cdot\hat{\mathbf{x}}+i\boldsymbol{\eta}\cdot\hat{\mathbf{p}}}\right)=\int d\mathbf{x}\,d\mathbf{p}\,e^{i\boldsymbol{\xi}\cdot\mathbf{x}+i\boldsymbol{\eta}\cdot\mathbf{p}}f(\mathbf{x},\mathbf{p})\,, \tag{3.1}$$

where $\hat{\rho}$ is a density matrix and Tr means the trace of a matrix. Thus one has

$$\begin{aligned} f(\mathbf{x},\mathbf{p}) &= \frac{1}{(2\pi)^4}\int d\boldsymbol{\xi}\,d\boldsymbol{\eta}\ \mathrm{Tr}\left(\hat{\rho}(\hat{\mathbf{x}},\hat{\mathbf{p}})e^{i\boldsymbol{\xi}\cdot\hat{\mathbf{x}}+i\boldsymbol{\eta}\cdot\hat{\mathbf{p}}}\right)e^{-i\boldsymbol{\xi}\cdot\mathbf{x}-i\boldsymbol{\eta}\cdot\mathbf{p}} \\ &= \frac{1}{(2\pi)^4}\int d\boldsymbol{\xi}\,d\boldsymbol{\eta}\int d\mathbf{q}\,d\mathbf{q}'\,\langle\mathbf{q}\mid\hat{\rho}\mid\mathbf{q}'\rangle\langle\mathbf{q}'\mid e^{i\boldsymbol{\xi}\cdot\hat{\mathbf{x}}+i\boldsymbol{\eta}\cdot\hat{\mathbf{p}}}\mid\mathbf{q}\rangle\,e^{-i\boldsymbol{\xi}\cdot\mathbf{x}-i\boldsymbol{\eta}\cdot\mathbf{p}}\,, \end{aligned} \tag{3.2}$$

With the aid of eqs. (2.8) and (2.9), one easily gets

$$\begin{aligned} f(\mathbf{x},\mathbf{p}) &= \frac{1}{(2\pi)^4}\int d\boldsymbol{\xi}d\boldsymbol{\eta}d\mathbf{q}\left\langle q_1+\frac{\eta_1\hbar}{2},\,q_2+\frac{\eta_2\hbar}{2}\right|\hat{\rho}\left|q_1-\frac{\eta_1\hbar}{2}-\frac{\xi_2\theta}{2},\,q_2-\frac{\eta_2\hbar}{2}+\frac{\xi_1\theta}{2}\right\rangle\times \\ &\quad\times e^{i\boldsymbol{\xi}\cdot(\mathbf{q}-\mathbf{x})}\,e^{-i\boldsymbol{\eta}\cdot\mathbf{p}}\,. \end{aligned} \tag{3.3}$$

After slight algebra one can rewrite the eq. (3.3) as

$$f(\mathbf{x},\mathbf{p})=\frac{1}{(2\pi)^2}\int d\boldsymbol{\eta}\left\langle x_1+\frac{\eta_1\hbar}{2},\,x_2+\frac{\eta_2\hbar}{2}\right|\hat{\rho}\left|x_1-\frac{\eta_1\hbar}{2}-\frac{i\theta}{2}\overleftarrow{\partial}_{x_2},\,x_2-\frac{\eta_2\hbar}{2}+\frac{i\theta}{2}\overleftarrow{\partial}_{x_1}\right\rangle e^{-i\boldsymbol{\eta}\cdot\mathbf{p}}\,. \tag{3.4}$$

For the case of pure state, by virtu of eq. (2.1), one can substitute $\hat{\rho}=\mid\psi_m\rangle\langle\psi_m\mid$ into eq. (3.4) and obtain

$$f_m(\mathbf{x},\mathbf{p})=\frac{1}{(2\pi)^2}\int d\boldsymbol{\eta}\,e^{-i\boldsymbol{\eta}\cdot\mathbf{p}}\,\psi_m\left(\mathbf{x}+\frac{\hbar}{2}\boldsymbol{\eta}\right)*_\theta\psi_m^*\left(\mathbf{x}-\frac{\hbar}{2}\boldsymbol{\eta}\right), \tag{3.5}$$

where ψ_ms are orthonormalized solutions of the Schrödinger equation (2.2) with the energy E_m. This is WF on the noncommutative space. Comparing it with the original form of WF given by [9]

$$f(\mathbf{x},\mathbf{p})=\frac{1}{(2\pi)^2}\int d\boldsymbol{\eta}\,e^{-i\boldsymbol{\eta}\cdot\mathbf{p}}\,\psi\left(\mathbf{x}+\frac{\hbar}{2}\boldsymbol{\eta}\right)\psi^*\left(\mathbf{x}-\frac{\hbar}{2}\boldsymbol{\eta}\right), \tag{3.6}$$

we find that the only difference between them is the ordinary product between the functions ψ and ψ^* in eq. (3.6) replaced by the $*_\theta$-product in eq. (3.5). This looks very simple and natural, since we are dealing with the matter on a noncommutative coordinate space. In the commuting space limit, $\theta\to 0$, eq. (3.5) would reduce to eq. (3.6). Here we would like to emphasize that the difference between eq. (3.5) and eq. (3.6) is of essential importance for building of deformation quantization of NCQM, because as we show it later the eq. (3.5) satisfies a generalized $*$-genvalue equation and the eq. (3.6) does not. This generalized $*$-genvalue equation plays an important role in the deformation quantization of NCQM as the Schrödinger equation does in ordinary quantum mechanics.

In fact, the WFs given by eq. (3.5) can be regarded as the diagonal elements of the density matrix (pure state), usually denoted as $f_m = f_{m,m}$. The general WFs on the non-commuting phase-space can be expressed as

$$f_{m,n}(\mathbf{x},\mathbf{p}) = \frac{1}{(2\pi)^2}\int d\boldsymbol{\eta}\, e^{-i\boldsymbol{\eta}\cdot\mathbf{p}}\, \psi_m\left(\mathbf{x}+\frac{\hbar}{2}\boldsymbol{\eta}\right) *_\theta \psi_n^*\left(\mathbf{x}-\frac{\hbar}{2}\boldsymbol{\eta}\right) = f^*_{n,m}(\mathbf{x},\mathbf{p}). \qquad (3.7)$$

As a consequence, matrix elements of operators are produced by mere phase-space integrals,

$$\langle \psi_n \mid \hat{\mathbf{A}} \mid \psi_m \rangle = (2\pi\hbar)^2 \int d\mathbf{x}d\mathbf{p}\, a(\mathbf{x},\mathbf{p})\, f_{m,n}(\mathbf{x},\mathbf{p}). \qquad (3.8)$$

We now show that the WF (eq. (3.7)) obeys the generalized $*$-genvalue equation. Without essential loss of generality, we consider $H(\mathbf{x},\mathbf{p}) = \frac{\mathbf{p}^2}{2m} + V(\mathbf{x})$. With the explicitly action of $*_\hbar$-product, we have

$$\begin{aligned} H(\mathbf{x},\mathbf{p}) * f_{m,n}(\mathbf{x},\mathbf{p}) \;=\; & \frac{1}{(2\pi)^2}\int d\boldsymbol{\eta} \left(\frac{1}{2m}\left(\left(i\partial_{\eta_1} - \frac{i\hbar}{2}\partial_{x_1}\right)^2 + \left(i\partial_{\eta_2} - \frac{i\hbar}{2}\partial_{x_2}\right)^2\right) + V(\mathbf{x})\right) \times \\ & \times *_\theta\; e^{-i\boldsymbol{\eta}\cdot(\mathbf{p}+\frac{i\hbar}{2}\overleftarrow{\boldsymbol{\partial}}_{\mathbf{x}})}\, \psi_m\left(\mathbf{x}+\frac{\hbar}{2}\boldsymbol{\eta}\right) *_\theta\; \psi_n^*\left(\mathbf{x}-\frac{\hbar}{2}\boldsymbol{\eta}\right). \end{aligned} \qquad (3.9)$$

Notice that in the right hand side of eq. (3.9), just the $*_\theta$-products are reserved. Making the integration by parts for the ∂_η and using the associativity of the $*_\theta$-product, we can write eq. (3.9) as

$$\begin{aligned} H(\mathbf{x},\mathbf{p}) * f(\mathbf{x},\mathbf{p}) \;=\; & \frac{1}{(2\pi)^2}\int d\boldsymbol{\eta}\, e^{-i\boldsymbol{\eta}\cdot\mathbf{p}}\left(\left(\frac{1}{2m}\left(\left(i\partial_{\eta_1} + \frac{i\hbar}{2}\partial_{x_1}\right)^2 + \left(i\partial_{\eta_2} + \frac{i\hbar}{2}\partial_{x_2}\right)^2\right) + \right.\right. \\ & \left.\left. + V\left(x_1 + \frac{\hbar}{2}\eta_1,\, x_2 + \frac{\hbar}{2}\eta_2\right)\right) *_\theta \psi_m(\mathbf{x}+\frac{\hbar}{2}\boldsymbol{\eta})\right) \times \\ & \times *_\theta \psi_n^*\left(\mathbf{x}-\frac{\hbar}{2}\boldsymbol{\eta}\right). \end{aligned} \qquad (3.10)$$

Also noticing the action of the effective differential operators $(i\partial_{\eta_1} + \frac{i\hbar}{2}\partial_{x_1})^2 + (i\partial_{\eta_2} + \frac{i\hbar}{2}\partial_{x_2})^2$ on ψ^* turns out to be null, and using eq. (2.2) we obtain

$$H(\mathbf{x},\mathbf{p}) * f_{m,n}(\mathbf{x},\mathbf{p}) \;=\; E_m\, f_{m,n}(\mathbf{x},\mathbf{p})\,. \qquad (3.11)$$

Likewise, we have

$$f_{m,n}(\mathbf{x},\mathbf{p}) * H(\mathbf{x},\mathbf{p}) \;=\; E_n\, f_{m,n}(\mathbf{x},\mathbf{p})\,. \qquad (3.12)$$

Here the key point of proving eq. (3.11) and eq. (3.12) relies on the fact that the product between ψ and ψ^* in the WF of eq. (3.7) is $*_\theta$-product, so the associativity of $*_\theta$-products guarantees that one can directly use eq. (2.2) to finish the proof. At this stage it is easy to understand why the original WF (eq. (3.6)) does not satisfy the generalized $*$-genvalue equation, it just satisfies the $*_\hbar$-genvalue equation $H *_\hbar f = f *_\hbar H = E\, f$.

Noticing the following properties of the $*$-product

$$\int d\mathbf{x}d\mathbf{p}\, f(\mathbf{x},\mathbf{p}) * g(\mathbf{x},\mathbf{p}) = \int d\mathbf{x}d\mathbf{p}\, g(\mathbf{x},\mathbf{p}) * f(\mathbf{x},\mathbf{p}) = \int d\mathbf{x}d\mathbf{p}\, f(\mathbf{x},\mathbf{p})\, g(\mathbf{x},\mathbf{p})\,, \quad (3.13)$$

one can see that the standard machinery of density matrices then is readily transcribed in this language, e.g. the trace relation

$$\int d\mathbf{x}d\mathbf{p} f_{m,n}(\mathbf{x},\mathbf{p}) = \int d\mathbf{x}\psi_m(\mathbf{x})\,\psi_n^*(\mathbf{x}) = \delta_{m,n}\,, \tag{3.14}$$

and

$$f_{m,n}(\mathbf{x},\mathbf{p}) * f_{l,k}(\mathbf{x},\mathbf{p}) = \frac{1}{(2\pi\hbar)^2}\delta_{n,l} f_{m,k}(\mathbf{x},\mathbf{p}). \tag{3.15}$$

From the above three equations it follows

$$\int d\mathbf{x}d\mathbf{p}\, f_{m,n}(\mathbf{x},\mathbf{p}) f_{k,l}^*(\mathbf{x},\mathbf{p}) = \frac{1}{(2\pi\hbar)^2}\delta_{m,k}\delta_{n,l}\,. \tag{3.16}$$

With the help of the following expression

$$\begin{aligned}\left\langle \mathbf{x}+\frac{\hbar}{2}\mathbf{y}\right| *_\theta \left|\mathbf{x}-\frac{\hbar}{2}\mathbf{y}\right\rangle &= e^{\frac{i\theta}{2}(\partial_{u_1}\partial_{v_2}-\partial_{u_2}\partial_{v_1})}\delta(u_1-v_1+\hbar y_1)\delta(u_2-v_2+\hbar y_2)\,|_{\mathbf{u}=\mathbf{v}=\mathbf{x}} \\ &= \delta(\hbar\mathbf{y})\,,\end{aligned} \tag{3.17}$$

because $(\partial_{u_1}\partial_{v_2}-\partial_{u_2}\partial_{v_1})\delta(u_1-v_1+\hbar y_1)\delta(u_2-v_2+\hbar y_2)=0$, one also has the resolution of the identity

$$\begin{aligned}\sum_n f_{n,n}(\mathbf{x},\mathbf{p}) &= \frac{1}{(2\pi)^2}\sum_n \int d\mathbf{y}\, e^{-i\mathbf{y}\cdot\mathbf{p}}\left\langle \mathbf{x}+\frac{\hbar}{2}\mathbf{y}|\psi_n\rangle\langle\psi_n\mid *_\theta|\mathbf{x}-\frac{\hbar}{2}\mathbf{y}\right\rangle \\ &= \frac{1}{(2\pi)^2}\int d\mathbf{y}\, e^{-i\mathbf{y}\cdot\mathbf{p}}\left\langle \mathbf{x}+\frac{\hbar}{2}\mathbf{y}\right| *_\theta \left|\mathbf{x}-\frac{\hbar}{2}\mathbf{y}\right\rangle = \frac{1}{(2\pi\hbar)^2} = \frac{1}{h^2}\,,\end{aligned} \tag{3.18}$$

Similarly, using completeness of the eigenvectors ψ_n of hamiltonian H, one also has

$$\sum_{m,n} f_{m,n}(\mathbf{x},\mathbf{p}) f_{m,n}^*(\mathbf{x}',\mathbf{p}') = \frac{1}{(2\pi\hbar)^2}\delta(\mathbf{x}-\mathbf{x}')\delta(\mathbf{p}-\mathbf{p}')\,. \tag{3.19}$$

Thus an arbitrary phase-space function $\varphi(\mathbf{x},\mathbf{p})$ can be expanded as

$$\varphi(\mathbf{x},\mathbf{p}) = \sum_{m,n} c_{m,n} f_{m,n}(\mathbf{x},\mathbf{p})\,, \tag{3.20}$$

the coefficients being specified through (3.16)

$$c_{m,n} = (2\pi\hbar)^2 \int d\mathbf{x}d\mathbf{p} f_{m,n}^*(\mathbf{x},\mathbf{p})\varphi(\mathbf{x},\mathbf{p})\,. \tag{3.21}$$

Thus we have a set of orthonormal and complete functions (WFs) to expand any function in the non-commuting phase-space.

4. Some examples

Now we consider a few of examples. First we discuss a free particle moving on the noncommutative plane with the hamiltonian operator $\hat{H} = \hat{\mathbf{p}}^2/2m$. This example is rather trivial. The solution of eq. (2.2) with a energy $E = \mathbf{p}_0^2/2m$ is a usual plane wave $\psi(\mathbf{x}) = e^{\frac{i}{\hbar}\mathbf{p}_0\cdot\mathbf{x}}$ (here the normalization factor is neglected). Substituting this plane wave into eq. (3.5), we have

$$f(\mathbf{x},\mathbf{p}) = \delta(\mathbf{p}_0 - \mathbf{p})\,. \tag{4.1}$$

Obviously this f satisfies

$$H * f = \frac{\mathbf{p}^2}{2m}\,\delta(\mathbf{p}_0 - \mathbf{p}) = E\,f\,. \tag{4.2}$$

Then we discuss a non-trivial example, i.e., the WF corresponding to ground state wave function of the 2-dimensional noncommutative harmonic oscillator with the hamiltonian operator $\hat{H} = \frac{1}{2}(\hat{\mathbf{p}}^2 + \hat{\mathbf{x}}^2)$. Here we take $\hbar$ and all parameters equal to 1 for convenience. In this case, the solution of eq. (2.2) was given in [16]. The ground state wave function is

$$\psi_0(\mathbf{x}) = \frac{1}{\sqrt[4]{\pi^2(1+\frac{\theta^2}{4})}}\,\exp\left(\frac{-\mathbf{x}^2}{2\sqrt{1+\frac{\theta^2}{4}}}\right) \tag{4.3}$$

and its energy is $E_0 = 1$. Substituting the ψ_0 into eq. (3.5), we have

$$\begin{aligned} f_0(\mathbf{x},\mathbf{p}) = {} & \frac{1}{4\pi^3\sqrt{1+\frac{\theta^2}{4}}}\int d\boldsymbol{\eta}\, e^{-i\boldsymbol{\eta}\cdot\mathbf{p}} \times \\ & \times \exp\left(\frac{-1}{2\sqrt{1+\frac{\theta^2}{4}}}\left(\left(x_1 + \frac{\eta_1}{2} + \frac{i\theta}{2}\partial_{x_2}\right)^2 + \left(x_2 + \frac{\eta_2}{2} - \frac{i\theta}{2}\partial_{x_1}\right)^2\right)\right) \times \\ & \times \exp\left(\frac{-1}{2\sqrt{1+\frac{\theta^2}{4}}}\left(\left(x_1 - \frac{\eta_1}{2}\right)^2 + \left(x_2 - \frac{\eta_2}{2}\right)^2\right)\right). \end{aligned} \tag{4.4}$$

Since usually thinking the noncommutative parameter θ very small, for simplicity of calculating, in the following calculation we just keep the first order of θ to show the effect of spatial noncommutativity, and obtain

$$f_0(\mathbf{x},\mathbf{p}) = \frac{1}{\pi^2}\, e^{-(\mathbf{x}^2+\mathbf{p}^2)}\,(1 - \theta(x_1p_2 - x_2p_1))\,. \tag{4.5}$$

After some algebra and up to the first order of θ, we have

$$\frac{1}{2}(\mathbf{x}^2+\mathbf{p}^2) * \frac{1}{\pi^2}\, e^{-(\mathbf{x}^2+\mathbf{p}^2)}\,(1 - \theta(x_1p_2 - x_2p_1)) = \frac{1}{\pi^2}\, e^{-(\mathbf{x}^2+\mathbf{p}^2)}\,(1 - \theta(x_1p_2 - x_2p_1))\,. \tag{4.6}$$

This means that f_0 given by eq. (3.5) indeed satisfies the generalized $*$-genvalue equation.

If one substitute the above ground state wave function ψ_0 into eq. (3.6), one should get a different expression

$$\tilde{f}_0(\mathbf{x},\mathbf{p}) = \frac{1}{\pi^2}\, e^{-(\mathbf{x}^2+\mathbf{p}^2)}\,. \tag{4.7}$$

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It is straightforward to check that the $\tilde{f}_0$ does not obey the generalized $*$-genvalue equation. So taking the form of eq. (3.6) as the WF on a noncommutative space seems not suitable, because it doesn't satisfy the corresponding $*$-genvalue equation.

It is well known that WF in the deformation quantization of ordinary quantum mechanics can be defined equivalently in two ways. One of them refers to the operator formulation of quantum mechanics according to eq. (3.6), and other is directly to solve the $*_\hbar$-genvalue equation and take its real solutions as the desired WF. It is reasonable to think that there are also two ways to define WF in the deformation quantization of NCQM. In this case, the $*_\hbar$-genvalue equation has been extended to $*$-genvalue equation (3.11) or (3.12) to reflect the characteristic of noncommutative space. Correspondingly, as we demonstrated above, eq. (3.6) should also be replaced by eq. (3.5) to give the WFs of NCQM.

5. Conclusion

In summary, we carefully investigate the fundamental aspects of deformation quantization in the non-commuting phase-space of NCQM in this paper. In order to match the basic commutation relations (1.1) of NCQM, we modify the expression of c-number phase-space kernels of Weyl ordered operators to reflect the spacial noncommutativity of this theory, and derive a new form for Wigner functions. Comparing our results with the original expressions, we find that the only change is replacing the ordinary product between specific functions in eq. (2.5) and eq. (3.6) by the $*_\theta$-product, as shown in eq. (2.7) and eq. (3.5). One can not use the properties (3.13) to eliminate the $*_\theta$-product in the integration expressions (2.7) and (3.5), because the $*_\theta$-product involves the partial derivatives for the phase-space variables x_i and the integration will be done for other variables. In addition, there exist other factors such as $e^{-i\mathbf{y}\cdot\mathbf{p}}$ in eq. (2.7) and eq. (3.5), which protect the $*_\theta$-product from various attempts to cancel it. Therefore, the expressions (2.7) and (3.5) are of essential difference in contrast to the ordinary expressions (2.5) and (3.6), respectively.

Upon modification of the c-number phase-space kernels, general relations of the Weyl correspondence in non-commuting phase-space have been properly changed to suit the above modification. WFs play an important role in the deformation quantization. Our new form of WFs satisfy the generalized $*$-genvalue equation and matrix elements of any operator can be calculated by mere integrating WFs and phase-space kernel of the operator. All these changes provide a consistent framework for construction of the deformation quantization of NCQM.

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